Supplementary derivations, benchmark policy, Milky Way gas-stress calculation, and claim audit

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1 Purpose and notation

This supplement gives the derivations and numerical policy used in the main text. The residual field is defined after the baryonic circular-speed contribution has been constructed,

$$V_b^2(R) = V_{\rm gas}^2(R) + V_{\rm disk}^2(R) + V_{\rm bulge}^2(R), \tag{1}$$

and the residual acceleration is

$$g_I(R) = \frac{V_{\rm obs}^2(R) - V_b^2(R)}{R}. \tag{2}$$

All benchmark models are evaluated against the same observed velocity vector and the same covariance prescription.

2 Axisymmetric finite-correlation response

Let the residual-field source be

$$\mathcal{J}_I(R) = \frac{2V_0^2}{\ell^2}\exp(-R^2/\ell^2). \tag{3}$$

The residual potential obeys

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{d\Phi_I}{dR}\right) = \mathcal{J}_I(R). \tag{4}$$

Multiplying by R and integrating from zero to R gives

$$R\frac{d\Phi_I}{dR} = \int_0^R R'\mathcal{J}_I(R')\,dR' = V_0^2\left[1-\exp(-R^2/\ell^2)\right]. \tag{5}$$

The residual circular-speed contribution is therefore

$$V_I^2(R) = R\frac{d\Phi_I}{dR} = V_0^2\left[1-\exp(-R^2/\ell^2)\right]. \tag{6}$$

The origin is regular, $V_I^2 \propto R^2$, and the large-radius limit is finite, $V_I^2 \to V_0^2$.

3 Least-informative source under finite moment constraints

Consider positive axisymmetric source densities with fixed normalization and fixed second moment,

$$\int_0^\infty 2\pi R \mathcal{J}(R) dR = J_0, \qquad \int_0^\infty 2\pi R^3 \mathcal{J}(R) dR = J_2. \tag{7}$$

The entropy functional

$$S = -\int_0^\infty 2\pi R \mathcal{J} \ln \mathcal{J} \, dR \tag{8}$$

has stationary variation

$$\delta \left[S - \lambda_0 \int 2\pi R \mathcal{J} dR - \lambda_2 \int 2\pi R^3 \mathcal{J} dR \right] = 0. \tag{9}$$

Thus

$$\ln \mathcal{J} + 1 + \lambda_0 + \lambda_2 R^2 = 0, \tag{10}$$

and

$$\mathcal{J}(R) = A \exp(-R^2/\ell^2). \tag{11}$$

This establishes the Gaussian as the least-informative finite-correlation source under the two stated constraints. It does not determine the physical value of ℓ; that value is fixed by the H I action closure.

4 Height-integrated H I dynamics

The starting point is the height-integrated gas system

$$\partial_t \Sigma + \nabla \cdot (\Sigma \boldsymbol{u}) = 0, \tag{12}$$

$$\Sigma(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u}) = -\nabla(\Sigma \sigma^2) - \Sigma \nabla(\Phi_b + \Phi_I). \tag{13}$$

Linear perturbations of a rotating disk lead to a stability structure governed by epicyclic frequency, surface density, and velocity dispersion. The Toomre form $Q = \sigma\kappa/(\pi G \Sigma)$ gives the local stability scale for gas-like perturbations, while the finite thickness and turbulent pressure smooth unresolved modes [1–3]. For the closure used here, the positive action measure is taken as

$$d\mu_A = \Sigma \sigma^2 dA, \tag{14}$$

which is the observable turbulent-pressure action carried by the H I layer. This measure is directly testable from moment-zero and moment-two maps.

5 Helmholtz decomposition and action reservoirs

The in-plane displacement field is decomposed as

$$\boldsymbol{\xi} = \nabla \chi + \hat{\boldsymbol{z}} \times \nabla \eta. \tag{15}$$

The first term is compressive and changes local surface density; the second is solenoidal and preserves area to first order. A finite-correlation separation assigns each sector to coherent and diffuse reservoirs. The four reservoirs are

$$\{C_{\mathrm{comp}}, C_{\mathrm{sol}}, D_{\mathrm{comp}}, D_{\mathrm{sol}}\}. \tag{16}$$

With only the total unresolved action specified and no preferred reservoir before radial projection, the constrained entropy

$$S_4 = -\sum_{j=1}^{4} p_j \ln p_j + \lambda \left(\sum_{j=1}^{4} p_j - 1 \right) \tag{17}$$

is stationary at

$$p_j = \frac{1}{4}. \tag{18}$$

The radial residual field is assigned to the coherent-compressive reservoir because that sector is the only reservoir that simultaneously changes surface density and remains phase coherent over a finite radial scale.

Proposition 1 (conditional H I action-quartile closure)**.** *Under the assumptions of a positive H I turbulent-action measure, Helmholtz-sector orthogonality, finite-correlation separation, and equal unresolved reservoir weights before radial projection, the residual-field correlation length is the radius enclosing one quarter of the total H I turbulent action:*

$$\int_{r \le \ell_I} \Sigma\sigma^2 dA = \frac{1}{4} \int \Sigma\sigma^2 dA. \tag{19}$$

If $\sigma^2 = \bar{\sigma}^2(1 + \epsilon h)$ *with* $|\epsilon h| \ll 1$*, then* $\ell_I = R_{25}^{\mathrm{HI}} + O(\epsilon)$.

Proof. The four-reservoir maximum-entropy stationary point gives $p_j = 1/4$. Coupling the radial residual response to the coherent-compressive reservoir selects one of the four reservoirs. The corresponding radial action radius is the radius enclosing one quarter of the H I action. If the dispersion field is slowly varying, the action cumulative is a smooth perturbation of the flux cumulative. The implicit-function theorem then gives a first-order displacement of the quartile radius proportional to the dispersion perturbation amplitude. □

6 Tail sensitivity of quartiles

Let $F(R)$ be a cumulative H I measure and R_q satisfy $F(R_q) = qF(\infty)$. Perturb the outer tail by ϵG with $G(R) = 0$ for $R < R_t$ and $R_q < R_t$. The first-order quantile displacement is

$$\delta R_q = \frac{q\delta F(\infty) - \delta F(R_q)}{F'(R_q)} = \frac{q\delta F(\infty)}{F'(R_q)}. \tag{20}$$

Outer-tail sensitivity is proportional to q for inner quantiles. The first quartile is therefore less sensitive to diffuse H I tails than the median or upper quartile, while remaining outside the beam-dominated central region in the resolved DDO154 map.

Table 1: DDO154 H I flux and action-proxy quartiles.

Measure	R_{25}	R_{50}	R_{75}	$R_{25}/R_{25}^{\mathrm{flux}}$
H I flux measure	2.638	5.472	9.177	1.000
H I turbulent action proxy: $\Sigma\sigma$	2.480	5.179	8.823	0.940
H I turbulent action proxy: $\Sigma\sigma^2$	2.366	4.938	8.538	0.897

7 Amplitude closure and acceleration-scale comparison

The acceleration scale is

$$a_H = \frac{cH_0}{2\pi} = 1.082 \times 10^{-10}\,\mathrm{m\,s^{-2}}. \tag{21}$$

It is close to the MOND/RAR acceleration scale [4–6]. The amplitude law

$$V_c^4 = GMa_H \tag{22}$$

therefore shares the baryonic Tully-Fisher scaling. The new test is not the scaling alone; it is the combination of this amplitude with a map-derived finite-correlation length.

Two masses are distinguished:

$$V_{c,\mathrm{gas}}^4 = GM_{\mathrm{gas}}a_H, \tag{23}$$

$$V_{c,\mathrm{bar}}^4 = G(M_{\mathrm{gas}} + M_\star)a_H. \tag{24}$$

The gas-only mass is suitable for a gas-dominated target such as DDO154. Stellar-dominated disks require the baryon-complete diagnostic.

Table 2: Amplitude diagnostics by target.

Galaxy	class	M_{gas} $M_\odot$	$M_\star$ $M_\odot$	$V_{c,\mathrm{gas}}$ km s^{-1}	$V_{c,\mathrm{bar}}$ km s^{-1}	$V_{I,\mathrm{diag}}$ km s^{-1}
DDO154	map-level	3.62×10^8	–	47.7	47.7	44.4
NGC3198	radial auxiliary	7.07×10^9	1.69×10^{10}	100.4	136.3	90.8
NGC7331	radial auxiliary	6.93×10^9	4.59×10^{10}	99.9	166.0	0.0

8 Radial covariance and nuisance marginalization

For a velocity residual vector $\Delta_i = V_{\mathrm{obs},i} - V_{\mathrm{mod},i}$, the likelihood is

$$\ln \mathcal{L} = -\frac{1}{2}\Delta^T C^{-1}\Delta - \frac{1}{2}\ln|C| - \frac{n}{2}\ln(2\pi), \tag{25}$$

with

$$C_{ij} = (1-\rho)\sigma_i^2\delta_{ij} + \rho\sigma_i\sigma_j \exp(-|R_i - R_j|/L_c). \tag{26}$$

The nuisance vector is

$$\theta_n = (s_D, s_i, \Upsilon_*, \rho, L_c, b), \tag{27}$$

representing distance, inclination-induced velocity normalization, stellar mass-to-light ratio, radial covariance, covariance length, and beam smearing. The same nuisance policy is used for QICT, NFW, Burkert, pseudo-isothermal, Einasto, and DC14-like families.

9 Halo benchmarks

The NFW model includes a weak concentration-mass prior. Einasto is evaluated with fixed shape parameter $\alpha = 0.18$ and fitted amplitude and scale radius. The DC14-like model uses a fixed feedback-modified core-cusp transition. Burkert and pseudo-isothermal profiles test cored halo shapes. These profiles are benchmark families evaluated under the same nuisance, covariance, and beam-smearing policy.

Table 3: Homogeneous nuisance marginalization summary.

galaxy	model	logZ	chi2_median	chi2_p16	chi2_p84
DDO154	QICT anchored	-21.2	5.4	2.98	10.7
DDO154	QICT posterior diagnostic	-21.3	5.56	3.16	12.2
DDO154	NFW c-M prior	-61.2	106	94.5	118
DDO154	Burkert	-22.7	8.91	5.79	14.2
NGC3198	QICT anchored	-210	591	388	783
NGC3198	QICT posterior diagnostic	-126	29.6	20.2	45.8
NGC3198	NFW c-M prior	-121	15.1	7.52	39.4
NGC3198	Burkert	-132	41.7	33	60
NGC7331	QICT anchored	-321	1.65e+03	1e+03	2.36e+03
NGC7331	QICT posterior diagnostic	-127	37.1	17.7	87.7
NGC7331	NFW c-M prior	-145	104	64.7	220
NGC7331	Burkert	-180	160	136	273

10 Milky Way gas-stress radial solver

The Milky Way ledger adds a component-resolved radial stress calculation. The active input files are `mw_radial_components.csv`, `mw_stress_components.csv`, `mw_rotation_curve.csv`, `mw_vertical_force_Kz.csv`, `mw_oort_constants.csv`, `mw_solar_geometry.json`, and `sgrA_prior.json`. The scoring curve is not used to build the response basis; it is used to estimate the non-negative channel amplitudes and to compute the benchmark metrics.

The component stress tensor has the schematic form

$$\Pi_b^{ij} = \sum_a \rho_a \left(\sigma_a^{ij} + \delta u_a^i \delta u_a^j + c_{s,a}^2 h^{ij} \right) + P_{\mathrm{fb}} h^{ij}, \tag{28}$$

with source acceleration

$$a_\Pi^i = (\rho_b + \epsilon)^{-1} \nabla_j \Pi_b^{ij}. \tag{29}$$

The radial implementation does not reconstruct the full tensor divergence; it evaluates a compact response basis:

$$B_1(R) = V_b^2(R), \tag{30}$$

$$B_2(R) = [\mathcal{G}_{\mathrm{H2}} V_{\mathrm{H2}}^2](R), \tag{31}$$

$$B_3(R) = V_0^2 \frac{R}{R + R_0/4}. \tag{32}$$

The molecular Green response uses the H_2 cumulative profile to define a compact molecular memory scale. The historical disk channel uses $R_0/4$ as a fixed coherence radius set by the Solar-geometry file. The black-hole copy-time gate uses the Sgr A* prior and is retained in the equations, but it is negligible on the supplied radial grid.

Let X be the matrix with columns B_1, B_2, and B_3, and let $y = V_{\mathrm{obs}}^2$. With velocity errors σ_V, the variance in y is approximated as $\sigma_y = 2V_{\mathrm{obs}}\sigma_V$. The response coefficients are obtained by

$$\hat{\boldsymbol{c}} = \arg\min_{\boldsymbol{c} \geq 0} \left\| \frac{X\boldsymbol{c} - y}{\sigma_y} \right\|_2^2. \tag{33}$$

The predicted circular speed is

$$V_{\mathrm{QICT}}(R) = \sqrt{X(R)\hat{\boldsymbol{c}}}. \tag{34}$$

The active coefficients are counted in AIC and BIC. Halo baselines are fitted on the same radial grid and the same velocity errors.

Table 4: Extended model comparison.

galaxy	model	BIC	RMS_kms	chi2
DDO154	Baryons only	824	22.7	824
DDO154	RAR fixed a0	81.7	6.74	81.7
DDO154	QICT anchored	2.21	0.839	2.21
DDO154	QICT posterior diagnostic	7.1	0.835	2.13
DDO154	NFW c-M prior	104	6.94	98.9
DDO154	Burkert	9.94	1.36	4.97
DDO154	Pseudo-isothermal	11.4	1.54	6.41
DDO154	Einasto alpha=0.18	23	2.64	18
DDO154	DC14-like core-cusp	12.8	1.74	7.81
NGC3198	Baryons only	553	40.8	553
NGC3198	RAR fixed a0	404	31.8	404
NGC3198	QICT anchored	553	40.8	553
NGC3198	QICT posterior diagnostic	28	4.41	20.2
NGC3198	NFW c-M prior	16.1	3.98	8.32
NGC3198	Burkert	40.7	5.68	32.9
NGC3198	Pseudo-isothermal	10.2	1.57	2.34
NGC3198	Einasto alpha=0.18	9.35	1.38	1.53
NGC3198	DC14-like core-cusp	39.3	5.68	31.5
NGC7331	Baryons only	1.6e+03	69.2	1.6e+03
NGC7331	RAR fixed a0	296	24.5	296
NGC7331	QICT anchored	1.6e+03	69.2	1.6e+03
NGC7331	QICT posterior diagnostic	22.5	4.28	14.6
NGC7331	NFW c-M prior	92.1	10.6	84.3
NGC7331	Burkert	155	11.8	147
NGC7331	Pseudo-isothermal	8.58	1.09	0.756
NGC7331	Einasto alpha=0.18	31.1	5.06	23.3
NGC7331	DC14-like core-cusp	159	11.9	151

The result is a component-resolved radial stress calculation. It is not an external-galaxy H I map-level validation and is therefore reported as a separate evidence class.

11 Map-level audit and ring-level stress test

The audit script requires a two-dimensional H I moment-zero FITS map for map-level status. A rotation curve, baryonic decomposition, or gas radial profile receives radial or ring-level status. The submitted files yield one map-level target, DDO154. NGC3198 and NGC7331 are radial auxiliary targets. The late-type archive is a ring-level stress test.

The ring-level analogue of the H I quartile is

$$\ell_{25}^{\rm gas} = Q_{0.25}(R; w), \qquad w(R) = \max[V_{\rm gas}(R)|V_{\rm gas}(R)|, 0]R\Delta R. \tag{35}$$

The retained sample contains 143 galaxies. The median ratio $\ell_{\rm post}/\ell_{25}^{\rm gas}$ is 0.583, and the anchored ring-level model beats the best halo benchmark in 27 galaxies. The result is a quantified population-level tension under a ring-level stress test.

12 Claim-status matrix

The analysis establishes a strict map-level result for DDO154. NGC3198, NGC7331, and the late-type ring-level sample retain radial or ring-level status because the submitted files lack the required H I

Table 5: Milky Way gas-stress model metrics.

Model	χ^2	χ^2/N	AIC	BIC
baryons only	18309.02	183.090	18309.02	18309.02
QICT gas-stress closure	149.24	1.492	155.24	163.05
Burkert + baryons	1464.51	14.645	1468.51	1473.72
NFW + baryons	1860.39	18.604	1864.39	1869.60

Table 6: Milky Way QICT channel amplitudes.

Response channel	Coefficient
direct baryonic inertia	0.12875
compact H2 memory	39.565
historical disk copy memory	0.49331

FITS maps for those cases. The posterior QICT diagnostic tests the flexibility of the finite-correlation functional form; its status is diagnostic rather than pre-scored. The gas-only amplitude is appropriate when the supplied baryonic mass is gas-dominated. Stellar-dominated systems require the baryon-complete diagnostic and, for map-level status, two-dimensional H I data.

13 Reproducibility

The package includes raw submitted archives, source tables, figure data, analysis scripts, LaTeX sources, compiled PDFs, the combined manuscript-plus-supplement PDF, and SHA256 hashes. The command `make reproduce` regenerates the reported numerical products from the included files. The author received no funding for this work and declares no competing interests.

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Table 7: Milky Way Oort diagnostic from the QICT radial field.

Quantity	Value
R_0 [kpc]	8.15
$V_c(R_0)$ [km s^{-1}]	230.05
$dV/dR\|_{R_0}$ [km s^{-1} kpc^{-1}]	-0.84
A [km s^{-1} kpc^{-1}]	14.53
B [km s^{-1} kpc^{-1}]	-13.70

Table 8: Ring-level stress-test summary.

Statistic	value
Retained galaxies	143
Median $\ell_{\mathrm{post}}/\ell_{25}^{\mathrm{gas}}$	0.583
MAD of $\ell_{\mathrm{post}}/\ell_{25}^{\mathrm{gas}}$	0.226
Within factor 2	75/143
Anchored QICT beats best halo	27/143
Posterior diagnostic beats best halo	64/143
Median ΔBIC anchored vs best halo	+12.37